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Differential Game of Pursuit Modelled by Infinite Three-Coupled System of Ordinary Differential Equations with Integral Constraints

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Abstract

This study examines a pursuit differential game of one pursuing player and one evading player modelled by an infinite three-coupled system of first-order ordinary differential equations. The control functions of the players adhere to integral constraints whereby the pursuing player has more control resources than the evading player. If, at some finite time, the pursuing player can drive the system's state from the initial state ξ^0 into the origin of the ℓ_2 space, the pursuit is then said to be completed. The evading player, however, aims to avert this from happening. We construct a control function and an admissible strategy for the pursuing player to solve the control problem and the differential game problem respectively. We give sufficient conditions for the pursuit to be completed in the game. In addition, we provide a concrete example to illustrate the application of our findings.

Keywords: admissible strategy; control problem; three-coupled; infinite system; integral constraint; pursuit differential game.

1 Introduction

This research contributes to the field of differential games, a branch of mathematical modelling focused on analyzing strategic interactions between rational participants, or players, over time, where the system's evolution is governed by differential equations. The foundational work of Isaacs [17] introduced key concepts and methodologies that have since been extensively developed, such as existence and characterization of values of differential games [5, 10], method of resolving functions in conflict-controlled processes [7], differential games of prescribed duration [12], convergence problem for differential games [13], pursuit differential games [15, 27], existence of equilibrium in differential games and the detailed analysis of optimal strategies [19], differential games theory in Distributed-Parameter Systems (DPS) [25], and simple pursuit by several objects [28]. For further reading on some foundational contributions to differential games and their classifications based on the number of players, we refer the interested reader to [38]. Differential games, particularly pursuit-evasion problems, have garnered significant attention due to their wide range of applications in defence [36, 16], robotics [26, 29], biological systems [22, 39], economics and management science [8], and other areas.

Many real-world systems, such as heat distribution [9], deformation of structures [11], and fluid flow [18], can be modelled as controlled systems using Partial Differential Equations (PDEs). This approach is particularly prevalent in modelling DPS, where state variables depend on both space and time. Moreover, recent contributions to the theory and application of model complex dynamical systems [20, 41] and impulsive differential equations in control systems [14, 40] have further enriched the methodological framework available for analyzing such systems. However, it is important to note that tackling PDEs and formulating controllers for these systems can pose significant challenges due to their inherent complexity. The control of systems governed by PDEs has been extensively studied, for instance, time-optimal control linked to the process of heat transfer [1], the time optimal control for evolutionary PDEs [4], adaptive control of PDEs [2, 35], control of DPS [3, 6], and many others.

A methodology of particular interest to us in this research is the decomposition method, which, when applied to a control system modelled by PDEs, reduces it to a control system modelled by an infinite system of ODEs. For instance, this method has been used for evolution-type DPS in Avdonin and Ivanov [3], and Satimov and Tukhtasinov [33] to obtain infinite systems of ODEs.

In many control systems, where finite resources such as fuel, energy, or materials are involved, the control parameters are typically limited by integral constraints. Such restrictions ensure that the total usage of these resources is capped over a given period, allowing the system to function effectively within its available capacity. Differential games in which players' control functions are subject to integral constraints have received significant attention; see, for example, the works of Salimi and Ferrara [30] and Sharifi et al. [34]. On the other hand, geometric constraints-arising from physical or structural limitations on the system's state and/or control variables-are also studied (for example, in Madhavan et al. [21] and Samatov et al. [31]).

In finite-dimensional differential games, the primary constraints on the players' control functions are integral, geometric, and mixed constraints [23]. Additionally, these constraints, to some extent, are retained for control problems modelled using an infinite system of ODEs. In this research, our interest lies more in differential games where the players' control functions are restricted by integral constraints.

In differential games, the conventional objective of the pursuing player is to complete pursuit in some way, while the evading player aims for the opposite. Research, where pursuit completion entails steering the system's state from a given initial state (different from the origin) into the origin of the state space, has received significant attention. Some works on differential games problems where pursuit completion entails steering the system's state into the origin of the Hilbert space include [21, 37] and many others.

Satimov and Tukhtasinov [32] studied differential games modelled by parabolic-type PDEs on the given interval $[0, T]$. The differential games were reduced to those modelled by the infinite system of ODEs,

$$\dot{z}_j = \gamma_j z_j - u_j + v_j, \qquad z_j(0) = z_j^0, \qquad j = 1, 2, \ldots, \tag{1}$$

where u_j and v_j, $j = 1, 2, \ldots$ are respectively the pursuing player's and the evading player's control parameters and the parameters γ_j, $j = 1, 2, \ldots$ satisfy:

$$0 > \gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \ldots \to -\infty. \tag{2}$$

Satimov and Tukhtasinov [32] analyzed four differential games with various control constraints and found sufficient conditions for completing pursuit and evasion. In each of the games studied, the pursuing player aims to steer the system's state from a specified initial state to the origin in a finite instant and the evading player aims to prevent this from happening.

Madhavan et al. [21] studied a differential game of pursuit whose mathematical model consists of the following infinite three-coupled system of ODE:

$$\begin{cases} \dot{x}_j = \mu_j x_j - u_{j1}(t) + v_{j1}(t), & x_j(0) = x_j^0, \\ \dot{y}_j = \lambda_j y_j - \gamma_j z_j - u_{j2}(t) + v_{j2}(t), & y_j(0) = y_j^0, \qquad j = 1, 2, \ldots, \\ \dot{z}_j = \gamma_j y_j + \lambda_j z_j - u_{j3}(t) + v_{j3}(t), & z_j(0) = z_j^0, \end{cases} \tag{3}$$

where $\mu_j, \lambda_j \leq 0$, $\gamma_j \in \mathbb{R}$, $u_{jk}, v_{jk} \in \mathbb{R}$ for $k = 1, 2, 3$, $x^0 = (x_1^0, x_2^0, \ldots) \in \ell_2$, $y^0 = (y_1^0, y_2^0, \ldots) \in \ell_2$, and $z^0 = (z_1^0, z_2^0, \ldots) \in \ell_2$, the pursuing and evading players' control parameters are respectively:

$$u = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, \ldots), \quad \text{and} \quad v = (v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, \ldots),$$

and they conform to integral constraints. The pursuing player in the game (3) strives to complete the pursuit by steering the state of the system (3) from the initial state into the origin of the ℓ_2 space within a finite time. They found sufficient conditions to solve the associated control problem and the differential game problem. However, the work of Madhavan et al. [21] did not consider the case in the game model (3) when the parameters μ_j, λ_j are positive for $j = 1, 2, \ldots$. Our work intends to fill this research gap.

The reviews mentioned above discuss various investigations into differential games modelled by infinite systems of two and three-coupled ODEs, with the pursuing and the evading players' control functions conforming to integral constraints and pursuit completion entailing guiding the system's state to the origin of the Hilbert space ℓ_2, a well-researched area. However, pursuit games may also occur in more intricate systems, including those governed by higher-order ODEs. Importantly, there is no universal solution formula for pursuit games in infinite systems since each system is characterized by its matrix-exponential, necessitating a tailored approach for each case.

This work examines a controlled system whose mathematical model consists of infinite three-coupled systems of ODE,

$$\begin{cases} \dot{x}_j = \mu_j x_j - u_{j1}(t) + v_{j1}(t), & x_j(0) = x_j^0, \\ \dot{y}_j = \lambda_j y_j - \gamma_j z_j - u_{j2}(t) + v_{j2}(t), & y_j(0) = y_j^0, \qquad j = 1, 2, \ldots, \\ \dot{z}_j = \gamma_j y_j + \lambda_j z_j - u_{j3}(t) + v_{j3}(t), & z_j(0) = z_j^0, \end{cases} \tag{4}$$

on the time interval $[0, \vartheta]$, where ϑ is sufficiently large, $0 \leq \mu_j \leq \mu$, $0 \leq \lambda_j \leq \lambda$, $\mu, \lambda, \gamma_j \in \mathbb{R}$, $u_{jk}, v_{jk} \in \mathbb{R}$ for $k = 1, 2, 3$. The setting of the game (4) is in the Hilbert space,

$$\ell_2 = \left\{ \xi = (\xi_1, \xi_2, \dots) \,:\, \sum_{j=1}^{\infty} |\xi_j|^2 < \infty,\ \xi_j = (x_j, y_j, z_j)^T,\ x_j, y_j, z_j \in \mathbb{R} \right\},$$

with inner product and norm respectively defined as:

$$\langle \xi, \eta \rangle = \sum_{j=1}^{\infty} \xi_j \eta_j, \quad \text{and} \quad \|\eta\| = \sqrt{\langle \eta, \eta \rangle} = \sqrt{\sum_{j=1}^{\infty} |\eta_j|^2} < \infty, \qquad \eta, \xi \in \ell_2.$$

The initial state, is $(x^0, y^0, z^0)^T = \xi^0 = (\xi_1^0, \xi_2^0, \dots)$ with $\xi_j^0 = (x_j^0, y_j^0, z_j^0)^T$, where we assume that $\xi^0 \in \ell_2$ is not the origin of the ℓ_2 space.

In the game (4), the pursuing player's and the evading player's control functions given respectively by:

$$u = (u_1, u_2, \dots) = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, \dots) \in \ell_2,$$
$$v = (v_1, v_2, \dots) = (v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, \dots) \in \ell_2,$$

conform to integral constraints.

To solve the control problem, we determine the control strategy that guides the controlled system from an initial state ξ^0 to the origin of the ℓ_2 space. Then, we analyse a pursuit game problem where the pursuing player strives to transfer the system's state back to the origin of ℓ_2 space in a finite duration, whereas the evading player aims to avoid this. We obtain sufficient conditions for the completion of pursuit in the game in a guaranteed pursuit time.

The remainder of the paper is structured in the following way: In Section 2, we present the mathematical model governing the player dynamics in the game and define some key terms. The presentation of results follows in Section 3, which consists of two subsections: Subsection 3.1 which deals with the solution of the control problem and Subsection 3.2 which is devoted to the solution of the pursuit differential game problem. In Section 4, we provide a concrete example to illustrate the application of our results and the paper is concluded in Section 5.

2 Problem Statement

Let $\varsigma > 0$, $\varrho_0 > \sigma_0 > 0$, and

$$\xi(t) = (\xi_1(t), \xi_2(t), \dots) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), \dots),$$
$$\xi_j(t) = (x_j(t), y_j(t), z_j(t))^T, \qquad |\xi_j(t)| = \sqrt{x_j^2(t) + y_j^2(t) + z_j^2(t)}, \qquad j = 1, 2, \dots,$$
$$\|\xi(t)\| = \sqrt{\sum_{j=1}^{\infty} x_j^2(t) + y_j^2(t) + z_j^2(t)}, \quad \|\xi^0\| = \sqrt{\sum_{j=1}^{\infty} (x_j^0)^2 + (y_j^0)^2 + (z_j^0)^2}.$$

Let $\omega(t)$, $t \in [0, \vartheta]$ be denoted by $\omega(\cdot)$.

The definitions below are vital for what comes next.

Definition 2.1. *The function,*

$$\omega[0,\vartheta]\to\ell_2,\quad \omega(\cdot)=(\omega_{11}(\cdot),\omega_{12}(\cdot),\omega_{13}(\cdot),\omega_{21}(\cdot),\omega_{22}(\cdot),\omega_{23}(\cdot),\dots)$$

is called an admissible control if the integral constraint,

$$\sum_{j=1}^{\infty}\int_0^{\vartheta}\left(\omega_{j1}^2(s)+\omega_{j2}^2(s)+\omega_{j3}^2(s)\right)ds\le\varsigma^2,\quad \varsigma>0,\qquad j=1,2,\dots,$$

is satisfied and its coordinates $\omega_{j1}(t),\omega_{j2}(t),\omega_{j3}(t)$, $t\in[0,\vartheta]$, *are measurable.*

We shall denote the set of all such admissible controls by $S(\varsigma)$.

Definition 2.2. *We refer to the functions* $u(\cdot)\in S(\varrho_0)$ *and* $v(\cdot)\in S(\sigma_0)$ *as the pursuing and the evading players' admissible controls, respectively.*

Definition 2.3. *An admissible strategy of the pursuing player is a function* $U:[0,\vartheta]\times\ell_2\to\ell_2$ *of the form:*

$$U(t,v)=(U_1(t,v),U_2(t,v),\dots),\quad U_j(t,v)=(U_{j1}(t,v),U_{j2}(t,v),U_{j3}(t,v)),$$

with components:

$$U_j(t,v(t))=v_j(t)-\omega_j(t),\quad v_j(t)=(v_{j1}(t),v_{j2}(t),v_{j3}(t)),\quad \omega_j(t)=(\omega_{j1}(t),\omega_{j2}(t),\omega_{j3}(t)),$$

satisfying, for every $v(\cdot)\in S(\sigma_0)$, *the constraint,*

$$\sqrt{\sum_{j=1}^{\infty}\int_0^{\vartheta}|U_j(t,v(t))|^2dt}\le\varrho_0,\quad |U_j(t,v(t))|^2=\sum_{k=1}^{3}U_{jk}^2(t,v(t)),$$

where $\omega(\cdot)=(\omega_1(\cdot),\omega_2(\cdot),\dots)\in S(\varrho_0-\sigma_0)$.

Definition 2.4. *In the game (4), we say that the time* $\vartheta>0$ *is a guaranteed time of pursuit if we can find a pursuing player strategy such that for any evading player admissible control, we have* $\xi(\theta)=0$ *at some time* θ, *where* $0\le\theta\le\vartheta$.

Define $C(0,\vartheta;\ell_2)$ as the space of continuous functions $\xi(t)\in\ell_2$, $t\in[0,\vartheta]$. Observe from [24] that if $0\le\mu_j\le\mu$, $0\le\lambda_j\le\lambda$ (where $\mu,\lambda\in\mathbb{R}^+$), $\gamma_j\in\mathbb{R}$ and $\omega(\cdot)\in S(\varsigma)$, then, a unique solution $\xi(t)=(\xi_1(t),\xi_2(t),\dots)$, $t\in[0,\vartheta]$ of the infinite three-coupled system of ODE,

$$\begin{cases}\dot{x}_j=\mu_jx_j+\omega_{j1}(t), & x_j(0)=x_j^0,\\ \dot{y}_j=\lambda_jy_j-\gamma_jz_j+\omega_{j2}(t), & y_j(0)=y_j^0,\\ \dot{z}_j=\gamma_jy_j+\lambda_jz_j+\omega_{j3}(t), & z_j(0)=z_j^0,\end{cases}\qquad j=1,2,\dots,\tag{5}$$

exists in the $C(0,\vartheta;\ell_2)$ space. Indeed, for $j=1,2,\dots$,

$$\xi_j(t)=\Phi_j(t)\xi_j^0+\int_0^t\Phi_j(t-s)\omega_j(s)ds,\tag{6}$$

where

$$\Phi_j(t)=\begin{pmatrix}e^{\mu_jt} & 0 & 0\\ 0 & e^{\lambda_jt}\cos\gamma_jt & -e^{\lambda_jt}\sin\gamma_jt\\ 0 & e^{\lambda_jt}\sin\gamma_jt & e^{\lambda_jt}\cos\gamma_jt\end{pmatrix},\qquad j=1,2,\dots.$$

The following property helps us to analyse the trajectory $\xi(t)$, $t\in[0,\vartheta]$ in the space ℓ_2.

Property 2.1. *It is easy to ascertain that* $\Phi_j(t)$ *satisfies:*

(a) $\Phi_j^{-1}(t) = \Phi_j(-t)$,

(b) $\Phi_j(t-s) = \Phi_j(t)\Phi_j(-s) = \Phi_j(-s)\Phi_j(t)$,

(c) $|\Phi_j(t)\xi_j| \le e^{\beta_j t}|\xi_j|$, *where* $\beta_j = \max\{\mu_j, \lambda_j\}$.

The research problems are contained in the following:

Problem 2.1. *Obtain sufficient conditions that resolve the control problem whose mathematical model is* (5).

Problem 2.2. *Obtain sufficient conditions that guarantee pursuit completion in the differential game modelled by* (4).

3 Results

In this section, we present the results of our research. In differential games of pursuit, the related control problem is typically analyzed before the differential game problem to establish a basis for developing a strategy that enables the pursuing player to effectively achieve pursuit completion.

3.1 The control problem

In this subsection, we address Problem 2.1. Here, we investigate an auxiliary control problem modelled by the infinite three-coupled system (5) that is useful in solving the pursuit problem. The goal is to find an admissible control $\omega(t)$ that can guide the system's state to the origin of ℓ_2 within a finite duration on the time interval $[0, \vartheta]$.

For $j = 1, 2, \ldots$, define the matrices $\chi_j(s) = \Phi_j(-s)\Phi_j^T(-s)$ and $\Upsilon_j(t) = \int_0^t \chi_j(s)ds$. A quick computation shows that,

$$\chi_j(s) = \begin{pmatrix} e^{-2\mu_j s} & 0 & 0 \\ 0 & e^{-2\lambda_j s} & 0 \\ 0 & 0 & e^{-2\lambda_j s} \end{pmatrix},$$

$$\Upsilon_j(t) = \begin{pmatrix} \int_0^t e^{-2\mu_j s}ds & 0 & 0 \\ 0 & \int_0^t e^{-2\lambda_j s}ds & 0 \\ 0 & 0 & \int_0^t e^{-2\lambda_j s}ds \end{pmatrix},$$

$$\Upsilon_j^{-1}(t) = \begin{pmatrix} \left(\int_0^t e^{-2\mu_j s}ds\right)^{-1} & 0 & 0 \\ 0 & \left(\int_0^t e^{-2\lambda_j s}ds\right)^{-1} & 0 \\ 0 & 0 & \left(\int_0^t e^{-2\lambda_j s}ds\right)^{-1} \end{pmatrix}.$$

The equation,

$$\sum_{j=1}^{\infty} (\xi_j^0)^T \Upsilon_j^{-1}(t)\xi_j^0 = \varsigma^2, \quad \varsigma > 0, \tag{7}$$

is crucial in solving Problem 2.1.

Theorem 3.1. *Let (7) have a root at $t = \vartheta_1$. Then, we can find an admissible control $\omega(\cdot) \in S(\varsigma)$ that can drive the state of (5) to the ℓ_2 space origin in a finite duration ϑ_1.*

Proof. We begin by designing the control for each $j = 1, 2, \dots$ as follows:

$$\omega_j(t) = \begin{cases} -\Phi_j^T(-t)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0, & 0 \le t \le \vartheta_1, \\ 0, & t > \vartheta_1. \end{cases} \tag{8}$$

Next, we establish the admissibility of our designed control (8). Observe that,

$$\begin{aligned} \sum_{j=1}^{\infty}\int_0^{\vartheta_1} |\omega_j(s)|^2\, ds &= \sum_{j=1}^{\infty}\int_0^{\vartheta_1} \left|-\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right|^2 ds, \\ &= \sum_{j=1}^{\infty}\int_0^{\vartheta_1} \left|\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right|^2 ds. \end{aligned} \tag{9}$$

But then,

$$\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0 = \begin{pmatrix} e^{-\mu_j s}\left(\int_0^{\vartheta_1} e^{-2\mu_j s} ds\right)^{-1} x_j^0 \\ e^{-\lambda_j s}\left(\int_0^{\vartheta_1} e^{-2\lambda_j s} ds\right)^{-1} \left(y_j^0 \cos\gamma_j s - z_j^0 \sin\gamma_j s\right) \\ e^{-\lambda_j s}\left(\int_0^{\vartheta_1} e^{-2\lambda_j s} ds\right)^{-1} \left(y_j^0 \sin\gamma_j s + z_j^0 \cos\gamma_j s\right) \end{pmatrix}. \tag{10}$$

Now, using (10) in (9), we get,

$$\begin{aligned} \sum_{j=1}^{\infty}\int_0^{\vartheta_1} |\omega_j(s)|^2\, ds &= \sum_{j=1}^{\infty}\int_0^{\vartheta_1} \left|\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right|^2 ds \\ &= \sum_{j=1}^{\infty}\int_0^{\vartheta_1} \Bigg\{ e^{-2\mu_j s}\left(\int_0^{\vartheta_1} e^{-2\mu_j s} ds\right)^{-2} (x_j^0)^2 \\ &\quad + e^{-2\lambda_j s}\left(\int_0^{\vartheta_1} e^{-2\lambda_j s} ds\right)^{-2} \left[(y_j^0)^2 + (z_j^0)^2\right] \Bigg\} ds \\ &= \sum_{j=1}^{\infty}\Bigg[(x_j^0)^2\left(\int_0^{\vartheta_1} e^{-2\mu_j s} ds\right)^{-1} + (y_j^0)^2\left(\int_0^{\vartheta_1} e^{-2\lambda_j s} ds\right)^{-1} \\ &\quad + (z_j^0)^2\left(\int_0^{\vartheta_1} e^{-2\lambda_j s} ds\right)^{-1} \Bigg] \\ &= \sum_{j=1}^{\infty} (\xi_j^0)^T \Upsilon_j^{-1}(\vartheta_1)\xi_j^0 \\ &= \varsigma^2. \end{aligned}$$

That is $\omega(\cdot) \in S(\varsigma)$. This shows that the designed control (8) conforms to integral constraint and thus, admissible.

Next, we show that the system's state (5) can be guided from ξ_j^0 to the ℓ_2 origin at the instant $t=\vartheta_1$ by utilizing the control (8), that is $\xi(\vartheta_1)=0$. Using Property 2.1(b) in (6) we get that,

$$\begin{aligned}\xi_j(\vartheta_1) &= \Phi_j(\vartheta_1)\xi_j^0 + \int_0^{\vartheta_1} \Phi_j(\vartheta_1)\Phi_j(-s)\omega_j(s)ds \\ &= \Phi_j(\vartheta_1)\left[\xi_j^0 + \int_0^{\vartheta_1} \Phi_j(-s)\omega_j(s)ds\right]. \end{aligned} \tag{11}$$

Deploying the control (8) in (11) gives:

$$\begin{aligned}\xi_j(\vartheta_1) &= \Phi_j(\vartheta_1)\left\{\xi_j^0 + \int_0^{\vartheta_1} \Phi_j(-s)\left[-\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right]ds\right\} \\ &= \Phi_j(\vartheta_1)\left\{\xi_j^0 - \int_0^{\vartheta_1} \Phi_j(-s)\Phi_j^T(-s)\left[\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right]ds\right\} \\ &= \Phi_j(\vartheta_1)\left\{\xi_j^0 - \left[\int_0^{\vartheta_1} \Phi_j(-s)\Phi_j^T(-s)ds\right]\left[\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right]\right\} \\ &= \Phi_j(\vartheta_1)\left\{\xi_j^0 - \Upsilon_j(\vartheta_1)\left[\Upsilon_j^{-1}(\vartheta_1)\xi_j^0\right]\right\} \\ &= \Phi_j(\vartheta_1)\left\{\xi_j^0 - \xi_j^0\right\} \\ &= 0.\end{aligned}$$

Thus, with the designed control (8), the state of (5) is transferred from the initial state ξ_j^0 into the ℓ_2 origin at the instant $t=\vartheta_1$. This completes the proof. □

Having resolved this initial stage, we can now focus on the pursuit game problem (4).

3.2 The pursuit differential game problem

In this subsection, we address Problem 2.2. We explore the pursuit differential game whose mathematical model is given by (4), which outlines the movement of a point in ℓ_2 space controlled by the pursuing and evading players with control parameters $u(\cdot)$ and $v(\cdot)$ respectively. We aim to fulfil the pursuing player's objective by devising an admissible strategy to bring the point to the origin at a certain time, regardless of the evading player's admissible control. The pursuit game is examined under integral constraints.

The game model has the solution $\xi_j(t)$, $j=1,2,\ldots$, given by:

$$\xi_j(t) = \Phi_j(t)\xi_j^0 + \int_0^t \Phi_j(t)\Phi_j(-s)\left[-U_j(s,v_j(s)) + v_j(s)\right]ds. \tag{12}$$

with the pursuing player's strategy embedded.

Consider the equation,

$$\sqrt{\sum_{j=1}^{\infty}(\xi_j^0)^T\Upsilon_j^{-1}(t)\xi_j^0} = \varrho_0 - \sigma_0, \quad \varrho_0,\sigma_0>0. \tag{13}$$

Theorem 3.2. *Let (13) have a solution at $t = \vartheta_2$. Then, a guaranteed pursuit time in the game (4) is ϑ_2.*

Proof. The steps in the proof are as follows:

Step 1: Devise a strategy for the pursuing player.
Step 2: Check that the developed strategy is admissible.
Step 3: Establish pursuit completion whenever the pursuing player deploys that strategy.

For Step 1, we offer to the pursuing player, the strategy below:

$$U_j(t, v_j(t)) = \begin{cases} v_j(t) + \Phi_j^T(-t)\Upsilon_j^{-1}(\vartheta_2)\xi_j^0, & 0 \le t \le \vartheta_2, \\ 0, & t > \vartheta_2, \end{cases} \tag{14}$$

where $v(\cdot) \in S(\sigma_0)$ is an admissible control of the evading player.

Next, for Step 2, we show that the strategy (14) is admissible; that is, that integral constraints are satisfied by (14). From (14),

$$\left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} |U_j(s, v_j(s))|^2\, ds\right)^{1/2} \le \left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} \left|v_j(s) + \Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_2)\xi_j^0\right|^2 ds\right)^{1/2}, \tag{15}$$

The RHS of (15) when estimated with Minkowskii's inequality gives:

$$\begin{aligned}\left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} |U_j(s, v_j(s))|^2\, ds\right)^{1/2} &\le \left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} |v_j(s)|^2\, ds\right)^{1/2} \\ &\quad + \left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} \left|\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_2)\xi_j^0\right|^2 ds\right)^{1/2} \\ &= \left(\sum_{j=1}^{\infty}\int_0^{\vartheta_2} |v_j(s)|^2\, ds\right)^{1/2} + \left(\sum_{j=1}^{\infty}(\xi_j^0)^T\Upsilon_j^{-1}(\vartheta_2)\xi_j^0\right)^{1/2} \\ &= \sigma_0 + \varrho_0 - \sigma_0 \\ &= \varrho_0.\end{aligned}$$

This verifies the admissibility of the pursuing player's strategy (14).

Lastly, for Step 3, we show that the pursuit can be completed whenever the pursuing player applies the strategy (14). Now, applying the strategy (14) in (12), we get that,

$$\begin{aligned}\xi_j(\vartheta_2) &= \Phi_j(\vartheta_2)\xi_j^0 + \int_0^{\vartheta_2} \Phi_j(\vartheta_2)\Phi_j(-s)\left[-U_j(s, v_j(s)) + v_j(s)\right] ds \\ &= \Phi_j(\vartheta_2)\xi_j^0 - \int_0^{\vartheta_2} \Phi_j(\vartheta_2)\Phi_j(-s)\Phi_j^T(-s)\Upsilon_j^{-1}(\vartheta_2)\xi_j^0 ds \\ &= \Phi_j(\vartheta_2)\left\{\xi_j^0 - \left[\int_0^{\vartheta_2} \Phi_j(-s)\Phi_j^T(-s) ds\right]\left[\Upsilon_j^{-1}(\vartheta_2)\xi_j^0\right]\right\} \\ &= \Phi_j(\vartheta_2)\left\{\xi_j^0 - \Upsilon_j(\vartheta_2)\Upsilon_j^{-1}(\vartheta_2)\xi_j^0\right\} \\ &= \Phi_j(\vartheta_2)\left[\xi_j^0 - \xi_j^0\right] \\ &= 0.\end{aligned}$$

Thus, a guaranteed pursuit time in the game is ϑ_2. This proves the theorem. □

4 Example

To illustrate an application of our results, we present the examples as follow:

Example 4.1. *Let us consider a differential game of pursuit modelled by (4) where the control parameters of the pursuing and the evading players adhere to integral constraints. In the game (4), suppose that* $\mu_j = \lambda_j = \frac{1}{2}$, $\varrho_0 = 4$, $\sigma_0 = 1$ *and that the initial state given is* $\xi^0 = (x^0, y^0, z^0)^T$ *where* $x^0 = y^0 = z^0 = \left(\frac{1}{j}\right)_{j=1}^{\infty} \equiv \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$. *Notice that* $x^0, y^0, z^0 \in \ell_2$ *since* $\sum_{j=1}^{\infty} \frac{1}{j^2} = \zeta(2) = \frac{\pi^2}{6} < \infty$, *where* $\zeta(\cdot)$ *is the Riemann zeta function. Substituting the given values in (13) yields:*

$$\sqrt{\sum_{j=1}^{\infty} \frac{3}{j^2}\left(\int_0^t e^{-s} ds\right)^{-1}} = 3,$$

which simplifies further into,

$$\frac{3}{1-e^{-t}} \sum_{j=1}^{\infty} \frac{1}{j^2} = 9. \tag{16}$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, *(16) yields:*

$$\begin{aligned} 3\left(\frac{\pi^2}{6}\right) &= 9(1-e^{-t}) \\ \implies e^t &= \frac{18}{18-\pi^2}. \end{aligned}$$

From the above, it is evident that,

$$t = \ln\left(\frac{18}{18-\pi^2}\right).$$

Thus, $\vartheta_2 = \ln\left(\frac{18}{18-\pi^2}\right)$. *Applying Theorem 3.2, we conclude that pursuit completion is attained for the initial values given, and the time* ϑ_2 *guarantees pursuit completion in the game.*

5 Discussion and Conclusions

In this study, we investigated control and differential game problems whose mathematical models are infinite three-coupled systems of ODE with integral constraints restricting the players' control functions. There are two players in the differential game problem studied: a pursuing player and an evading player. Although the control problem is used to solve the differential game issue, it is nevertheless important.

It is important to note that in (7) and (13), the matrix $\Upsilon_j(t)$ (for $t > 0$) and its inverse are positive definite. Moreover, $\Upsilon_j^{-1}(t)$ is strictly decreasing with respect to t in the sense of Loewner order for each j; consequently, the quadratic form $(\xi_j^0)^T \Upsilon_j^{-1}(t)\xi_j^0$ is strictly decreasing in t for each j. Furthermore, since the sequences (μ_j) and (λ_j) are bounded and nonnegative and $\xi^0 \in \ell_2$, the series,

$$f(t) = \sum_{j=1}^{\infty} (\xi_j^0)^T \Upsilon_j^{-1}(t)\xi_j^0,$$

converges for each fixed $t > 0$. In fact, $f(t)$ is a continuous and strictly decreasing function of t on $(0, \infty)$; it diverges to $+\infty$ as $t \to 0^+$, and converges to a finite positive limit as $t \to \infty$. Thus, by the Intermediate Value Theorem, if

$$\lim_{t\to\infty} f(t) < \varsigma^2, \quad \text{(respectively, } \lim_{t\to\infty} f(t) < (\varrho_0 - \sigma_0)^2\text{)},$$

then (7), (respectively, (13)) admits a unique root at $t = \vartheta_1$ (respectively, $t = \vartheta_2$).

We made the following contributions. We found sufficient conditions for the system's state to be guided into the ℓ_2 origin and have resolved the control problem associated with the system. Furthermore, we devised an admissible strategy for the pursuing player and demonstrated pursuit completion at a guaranteed time of pursuit. A concrete example that illustrates the application of our theorem in establishing pursuit completion at a guaranteed pursuit time is also given in Example 4.1.

Madhavan et al. [21] studied a differential game problem whose mathematical model consists of an infinite three-coupled system of ODEs (3) for the case $\mu_j, \lambda_j \leq 0$, and constructed a pursuit completion strategy for the pursuer which exists under the classical condition $\varrho_0 > \sigma_0$. However, our research has, for the first time, studied the differential game (3) for the case where μ_j, λ_j are positive, and we constructed a pursuit completion strategy, given by (14), which exists if $\varrho_0 > \sigma_0 + \sqrt{\lim_{t\to\infty} f(t)}$.

Future works could explore differential games of evasion modelled by the infinite three-coupled system (4). To make things more interesting, one could also study the differential game (4) for the case where mixed constraints are prescribed on the players' control parameters.

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References

[1] S. Albeverio & S. Alimov (2008). On a time-optimal control problem associated with the heat exchange process. *Applied Mathematics and Optimization*, *57*, 58–68. https://doi.org/10.1007/s00245-007-9008-7.

[2] H. Anfinsen & O. M. Aamo (2019). *Adaptive Control of Hyperbolic PDEs*. Communications and Control Engineering. Springer Nature, Cham, Switzerland. https://doi.org/10.1007/978-3-030-05879-1.

[3] S. A. Avdonin & S. A. Ivanov (1995). *Families of Exponentials: The Method of Moments In Controllability Problems For Distributed Parameter Systems*. Cambridge University Press, Melbourne, Australia.

[4] A. A. Azamov & M. B. Ruziboyev (2013). The time-optimal problem for evolutionary partial differential equations. *Journal of Applied Mathematics and Mechanics*, *77*(2), 220–224. https://doi.org/10.1016/j.jappmathmech.2013.07.013.

[5] L. D. Berkovitz (1988). Characterizations of the values of differential games. *Applied Mathematics and Optimization*, *17*(1), 177–183. https://doi.org/10.1007/BF01448365.

[6] F. L. Chernous'ko (1992). Bounded controls in distributed-parameter systems. *Journal of Applied Mathematics and Mechanics*, *56*(5), 707–723. https://doi.org/10.1016/0021-8928(92)90057-F.

[7] A. Chikrii (2013). *Conflict-Controlled Processes*. Springer Science & Business Media, Dordrecht. https://doi.org/10.1007/978-94-017-1135-7.

[8] E. Dockner (2000). *Differential Games In Economics and Management Science*. Cambridge University Press, Melbourne, Australia. https://doi.org/10.1017/CBO9780511805127.

[9] S. Dubljevic, J. P. Humaloja & M. Kurula (2022). Explicit model predictive control for PDEs: The case of a heat equation. *IFAC-PapersOnLine*, *55*(30), 460–465. https://doi.org/10.1016/j.ifacol.2022.11.096.

[10] R. J. Elliott & N. J. Kalton (1972). *The Existence of Value in Differential Games*. American Mathematical Society, Providence, Rhode Island.

[11] J. Fang, Z. Xiao, X. Zhu, L. You, X. Wang & J. Zhang (2024). Fast and compact partial differential equation (PDE)-based dynamic reconstruction of extended position-based dynamics (XPBD) deformation simulation. *Mathematics*, *12*(20), Article ID: 3175. https://doi.org/10.3390/math12203175.

[12] W. H. Fleming (1957). A note on differential games of prescribed duration. In *Contributions to The Theory of Games*, volume 39 of *Annals of Mathematics Studies* pp. 407–416. Princeton University Press, Princeton, NJ. https://doi.org/10.1515/9781400882151-024.

[13] W. H. Fleming (1961). The convergence problem for differential games. *Journal of Mathematical Analysis and Applications*, *3*(1), 102–116. https://doi.org/10.1016/0022-247X(61)90009-9.

[14] Y. Guo, X. B. Shu, F. Xu & C. Yang (2024). HJB equation for optimal control system with random impulses. *Optimization*, *73*(4), 1303–1327. https://doi.org/10.1080/02331934.2022.2154607.

[15] O. Hájek (1975). *Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion*. Mathematics In Science and Engineering. Academic Press, New York.

[16] E. Ho, A. Rajagopalan, A. Skvortsov, S. Arulampalam & M. Piraveenan (2022). Game theory in defence applications: A review. *Sensors*, *22*(3), Article ID: 1032. https://doi.org/10.3390/s22031032.

[17] R. Isaacs (1965). *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*. Dover Publications, Mineola, New York.

[18] M. Jokar, H. Salarieh & H. N. Pishkenari (2022). Satellite attitude PDE-based controller design considering fluid fuel sloshing in planar maneuver. *Aerospace Science and Technology*, *122*, Article ID: 107366. https://doi.org/10.1016/j.ast.2022.107366.

[19] N. N. Krasovskij, A. I. Subbotin & S. Kotz (1988). *Game-Theoretical Control Problems*. Springer, New York.

[20] G. S. Kumar & C. Gunasundari (2023). Dynamical analysis of two-preys and one predator interaction model with an Allee effect on predator. *Malaysian Journal of Mathematical Sciences*, *17*(3), 263–281. https://doi.org/10.47836/mjms.17.3.03.

[21] D. N. Madhavan, I. A. Alias, G. Ibragimov & R. M. Hasim (2024). Some results on pursuit games for an infinite system of ternary differential equations. *Malaysian Journal of Mathematical Sciences*, *18*(3), 567–581. https://doi.org/10.47836/mjms.18.3.07.

[22] Z. Mu, J. Pan, Z. Zhou, J. Yu & L. Cao (2023). A survey of the pursuit–evasion problem in swarm intelligence. *Frontiers of Information Technology & Electronic Engineering*, *24*(8), 1093–1116. https://doi.org/10.1631/FITEE.2200590.

[23] M. S. Nikolskii (1969). The direct method in linear differential games with integral constraints. *Controlled Systems*, *2*, 49–59.

[24] C. S. Odiliobi, R. M. Hasim & G. Ibragimov (2024). Existence of a unique solution of an infinite three-coupled system model of ordinary differential equations in Hilbert space. *Menemui Matematik (Discovering Mathematics)*, *46*(1), 17–28.

[25] Y. S. Osipov (1975). On the theory of differential games in systems with distributed parameters. *Doklady Akademii Nauk SSSR*, *223*(6), 1314–1317.

[26] P. Penar & Z. Hendzel (2022). Experimental verification of the differential games and H_∞ theory in tracking control of a wheeled mobile robot. *Journal of Intelligent & Robotic Systems*, *104*(4), Article ID: 61. https://doi.org/10.1007/s10846-022-01584-6.

[27] L. A. Petrosjan (1993). *Differential Games of Pursuit*. Optimization: Volume 2. World Scientific, Singapore. https://doi.org/10.1142/1670.

[28] B. N. Pshenichnyi (1976). Simple pursuit by several objects. *Cybernetics*, *12*(3), 484–485. https://doi.org/10.1007/BF01070036.

[29] U. Ruiz (2024). Surveillance evasion between two identical differential drive robots. *European Journal of Control*, *75*, Article ID: 100935. https://doi.org/10.1016/j.ejcon.2023.100935.

[30] M. Salimi & M. Ferrara (2019). Differential game of optimal pursuit of one evader by many pursuers. *International Journal of Game Theory*, *48*(2), 481–490. https://doi.org/10.1007/s00182-018-0638-6.

[31] B. T. Samatov, M. A. Horilov & A. A. Akbarov (2022). Differential game: "Life line" for non-stationary geometric constraints on controls. *Lobachevskii Journal of Mathematics*, *43*(1), 237–248. https://doi.org/10.1134/S1995080222040187.

[32] N. Y. Satimov & M. Tukhtasinov (2006). Game problems on a fixed interval in controlled first-order evolution equations. *Mathematical Notes*, *80*, 578–589. https://doi.org/10.1007/s11006-006-0177-5.

[33] N. Y. Satimov & M. Tukhtasinov (2007). On game problems for second-order evolution equations. *Russian Mathematics*, *51*(1), 49–57. https://doi.org/10.3103/S1066369X07010070.

[34] S. Sharifi, A. J. Badakaya & M. Salimi (2022). On game value for a pursuit-evasion differential game with state and integral constraints. *Japan Journal of Industrial and Applied Mathematics*, *39*(2), 653–668. https://doi.org/10.1007/s13160-022-00501-6.

[35] A. Smyshlyaev & M. Krstic (2010). *Adaptive Control of Parabolic PDEs*. Princeton University Press, Princeton, New Jersey. https://doi.org/10.1515/9781400835362.

[36] B. Tong, H. Duan, J. Xia & X. Liu (2022). Multi-agent border defense differential game based on biological predation and escape behavior in an obstacle environment. *Science China-Information Sciences*, *52*(12), 2213–2224. https://doi.org/10.1360/SSI-2022-0211.

[37] M. Tukhtasinov (1995). Some problems in the theory of differential pursuit games in systems with distributed parameters. *Journal of Applied Mathematics and Mechanics*, *59*(6), 935–940. https://doi.org/10.1016/0021-8928(95)00126-3.

[38] I. E. Weintraub, M. Pachter & E. Garcia (2020). An introduction to pursuit-evasion differential games. In *2020 American Control Conference (ACC)*, pp. 1049–1066. IEEE, Denver, CO, USA. https://doi.org/10.23919/ACC45564.2020.9147205.

[39] C. Xu, Y. Zhang, W. Wang & L. Dong (2022). Pursuit and evasion strategy of a differential game based on deep reinforcement learning. *Frontiers in Bioengineering and Biotechnology*, *10*, Article ID: 827408. https://doi.org/10.3389/fbioe.2022.827408.

[40] Q. B. Yin, X. B. Shu, Y. Guo & Z. Y. Wang (2024). Optimal control of stochastic differential equations with random impulses and the Hamilton–Jacobi–Bellman equation. *Optimal Control Applications and Methods*, *45*(5), 2113–2135. https://doi.org/10.1002/oca.3139.

[41] B. Zhou, X. B. Shu, F. Xu, F. Yang & Y. Wang (2024). Exponential synchronization of dynamical complex networks via random impulsive scheme. *Nonlinear Analysis: Modelling and Control*, *29*(4), 816–832. https://doi.org/10.15388/namc.2024.29.35728.